Solution to exercise in Tut 1 & 2.

Reminder: Here, the computation resule might exist some typos, you can send an email to me to point out.
What's more important for you is to fully understand the work flow. the idea behind each step!!!

Tut 1

"Linear."

1. $y' - 2ty = t, \quad y(0) = 1.$

Ans: $u(t) = e^{-\int_0^t 2s\,ds} = e^{-t^2}$

$$y(t) = \frac{\int_0^t s \cdot e^{-s^2} ds + C}{e^{-t^2}} = \frac{\frac{1}{2}(1 - e^{-t^2}) + C}{e^{-t^2}}$$

$y(0) = C = 1.$

$y(t) = \frac{3}{2} e^{t^2} - \frac{1}{2}$

2. $y' - \frac{2t}{1+t^2} y = 1, \quad y(1) = 2$

Ans: $u(t) = e^{-\int_1^t \frac{2s}{1+s^2} ds} = e^{-\ln(1+s^2)\Big|_{s=1}^{s=t}} = \frac{2}{1+t^2}$

$$y(t) = \frac{\int_1^t 1 \cdot \frac{2}{1+s^2} ds + C}{\frac{2}{1+t^2}}$$

$$\int_1^t \frac{1}{1+s^2} ds = \int_{\tan^{-1}(1)}^{\tan^{-1}(t)} \frac{1}{1+\tan(x)^2} \cdot d(\tan(x)) \quad (s = \tan(x))$$

$$= \int_{\tan^{-1}(1)}^{\tan^{-1}(t)} \cos^2(x) \tan'(x)\, dx$$

$$= \tan^{-1}(t) - \frac{\pi}{4}$$

$$y(t) = \frac{2 \cdot \tan^{-1}(t) - \frac{\pi}{2} + C}{\frac{2}{1+t^2}}, \quad \text{with } y(1) = 2$$

then $C = 2$

3. $y' = 1 + x + y + xy \quad . \quad y(0) = 0$

Ans:

$$y' - (1+x)y = 1+x.$$

$$u(x) = e^{-\int_0^x (1+s)\,ds} = e^{-(x+\frac{x^2}{2})}$$

$$y = \frac{\int_0^x (1+s)\cdot e^{-(s+\frac{s^2}{2})}\,ds + C}{e^{-(x+\frac{x^2}{2})}} = \left[e^{-(s+\frac{s^2}{2})}\Big|_{s=x}^{s=0} + C \right] \cdot e^{(x+\frac{x^2}{2})}$$

$$= \quad C + e^{(x+\frac{x^2}{2})} - 1$$

$$y(0) = 0 \Rightarrow C = 0.$$

You can solve the problem by separable variable method.

$$y' = (1+x)(1+y). \qquad \frac{1}{1+y}\,dy = (1+x)\,dx$$

$$\ln(1+y) = x + \frac{x^2}{2} + C$$

$$y = C' \cdot \exp(x + \frac{x^2}{2}) - 1.$$

$$y(0) = 0 \Rightarrow C' = 1$$

"Separable"

1、 $y' = 1 + \frac{y}{t} + (\frac{y}{t})^2 . \quad t > 0, \quad y(1) = 1$

Ans: $V(t) := \frac{y(t)}{t}$ then $V' = \frac{1+V^2}{t}.$

$$\frac{1}{1+V^2}\,dV = \frac{1}{t}\,dt. \qquad \text{so} \quad \tan^{-1}(V(t)) = \ln(t) + C$$

$$V(t) = \tan(\ln(t) + C).$$

$$y(1) = V(1)\cdot 1 = \tan(C) = 1. \qquad \Rightarrow C = \frac{\pi}{4}$$

2. $y' = \dfrac{y^2 - ty + t^2}{t^2}$ $\quad y(1) = 2$

Ans: $v(t) := \dfrac{y(t)}{t}$, $\quad (t \cdot v)' = v^2 - v + 1 = g(v)$. $\quad v(1) = 2$.

$$v' = \frac{g(v) - v}{t} = \frac{(v-1)^2}{t}.$$

$$\frac{1}{(v-1)^2} dv = \frac{1}{t} dt.$$

$$\frac{-1}{v-1} = \ln(t) + C. \quad , \quad v(t) = \frac{-1}{\ln(t) + C} + 1$$

$v(1) = 2 \Rightarrow C = -1$. $\quad$ so $v(t) = 1 - \dfrac{1}{\ln(t) - 1}$

$$y(t) = t - \frac{t}{\ln(t) - 1}.$$

"Exact".

1. $(y \cdot \cos(t) + 2te^y) dt + (\sin(t) + t^2 e^y + 2) dy = 0$. $\quad y(0) = 1$

Ans: $h(t, y) = \int (y \cdot \cos(t) + 2te^y) dt + C(y)$

$$= y \cdot \sin(t) + t^2 \cdot e^y + C(y)$$

$$h(t,y) = \int (\sin(t) + t^2 e^y + 2) dy + D(t)$$

$$= y \cdot \sin(t) + t^2 \cdot e^y + 2y + D(t).$$

So a choice for $C(y)$, $D(t)$ is

$$C(y) = 2y, \quad D(t) = 0$$

$\therefore \; y \cdot \sin(t) + t^2 \cdot e^y + 2y = E.$

with $y(0)=1$. we have

$$1\cdot \sin(0) + 0\cdot e^{1} + 2 = 2 = E.$$

2. $2ty\,dt + (2y+t^2)\,dy = 0$, $y(1)=2$

Ans: $h(t,y) = \int 2ty\,dt + C(y) = t^2y + C(y)$

$$= \int (2y+t^2)\,dy + D(t) = y^2 + t^2y + D(t)$$

$$\Rightarrow \begin{cases} C(y) = y^2 \\ D(t) = 0 \end{cases}$$

$$t^2y + y^2 = E. \qquad E = 6.$$

"Bernouli"

$y' = 5y - 5ty^3$, $y(0) = \frac{1}{10}$

Ans: $v = y^{-2}$.

$$v' = -2y^{-3}\cdot y' = -2y^{-3}(5y - 5ty^3)$$

$$= -10v + 10t$$

$$v' + 10v = 10t$$

$$u(t) = e^{\int_0^t 10\,ds} = e^{10t}$$

$$v = \frac{\int_0^t (e^{10s}\cdot 10s)\,ds + C}{e^{10t}}$$

with $v(0) = y(0)^{-2} = 100$.

$$C = 100.$$

$$\therefore v(t) = \left[\frac{1}{10}\int_0^{10t} e^{s}\cdot s\,ds + 100\right] e^{-10t}$$

$$= \left[\frac{1}{10}(s-1)\cdot e^{s}\Big|_{s=0}^{s=10t} + 100\right] e^{-10t}$$

$$= \frac{t}{10} + \frac{1}{10}e^{-10t} - 1 + 100$$

2. $y' + ty = ty^4$. $y(0) = 2$

Ans: $v = y^{-3}$. $v' = -3y^{-4} \cdot y' = -3y^{-4}(-ty + ty^4) = 3ty^{-3} - 3t$

$$= 3tv - 3t.$$

$$v' - 3tv = -3t.$$

$$u(t) = e^{-\int_0^t 3s\,ds} = e^{-\frac{3}{2}t^2}$$

$$v(t) = \frac{\int_0^t -3s e^{-\frac{3}{2}s^2}\,ds + C}{e^{-\frac{3}{2}t^2}}$$

$$= (C-1) \cdot e^{\frac{3}{2}t^2} + 1$$

$$v(0) = 2^{-3}, \Rightarrow C = 2^{-3}$$

"constant coefficient"

1. $y'' + 4y = \sin(x)$

Ans: fundamental solution for $y'' + 4y = 0$ is

$$\{\cos(2x), \sin(2x)\}.$$

And we may guess a special solution to

$y'' + 4y = \sin(x)$ is $y_{sp} = k \cdot \sin(x)$.

$y'' + 4y = (-k + 4k)\sin(x) = \sin(x)$

$\therefore$ general solution is

$$A \cdot \sin(2x) + B \cdot \cos(2x) + \frac{1}{3}\sin(x)$$

2. $y'' + 2y' + 4y = e^x$.

Ans: roots of $r^2 + 2r + 4 = 0$ are

$$-1 \pm \sqrt{3}.$$

fundamental solutions are

$$e^{-x} \cdot \cos(\sqrt{3}x), \quad e^{-x} \cdot \sin(\sqrt{3}x)$$

a special solution to $y'' + 2y' + 4y = e^x$

is $y_{sp} = k \cdot e^x$, then $7k = 1$. $k = \frac{1}{7}$.

general solution:

$$[A \cdot \cos(\sqrt{3}x) + B \cdot \sin(\sqrt{3}x)]e^{-x} + \frac{1}{7}e^x$$

3. $y'' - 2y' + y = \frac{e^x}{x^2+1}$

Ans: fundamental solution to $y'' - 2y' + y = 0$ is

$$y_1 = e^x, \quad y_2 = e^x \cdot x.$$

$$W(y_1, y_2) = y_1 \cdot y_2' - y_1' \cdot y_2$$
$$= e^{2x}.$$

By Theorem of Variation of Parameter.

a special solution is

$$L(x) = -e^x \cdot \int \frac{x \cdot e^x \cdot e^x}{(x^2+1) \cdot e^{2x}} dx + x \cdot e^x \cdot \int \frac{1}{(x^2+1)} dx$$

$$= -e^x \cdot \left(\frac{1}{2} \ln(x^2+1)\right) + x \cdot e^x \cdot \tan^{-1}(x)$$

Solution to Tut 2.

"Infinite Sum"

1. $f(x) = x^2, \quad -\frac{1}{2} \le x \le \frac{1}{2}$

$$a_0 = 1 \int_{-\frac{1}{2}}^{\frac{1}{2}} x^2 dx = \frac{x^3}{3} \Big|_{-\frac{1}{2}}^{\frac{1}{2}} = \frac{1}{12}$$

$$a_n = 2 \int_{-\frac{1}{2}}^{\frac{1}{2}} x^2 \cdot \cos\left(\frac{2\pi}{1} \cdot n x\right) dx = 2 \cdot \left[x^2 \cdot \frac{1}{2\pi n} \sin(2\pi n x) \Big|_{-\frac{1}{2}}^{\frac{1}{2}} - \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{x}{\pi n} \cdot \sin(2\pi n x)\, dx \right]$$

$$= \frac{2}{\pi n} \cdot \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{x}{2\pi n} [\cos(2\pi n x)]' dx$$

$$= \frac{2}{\pi n} \cdot \left[\frac{x}{2\pi n} \cdot \cos(2\pi n x) \Big|_{-\frac{1}{2}}^{\frac{1}{2}} - \frac{1}{2\pi n} \int_{-\frac{1}{2}}^{\frac{1}{2}} \cos(2\pi n x) dx \right]$$

$$= \frac{1}{(\pi n)^2} (-1)^n$$

$$b_n = 2 \cdot \int_{-\frac{1}{2}}^{\frac{1}{2}} x^2 \cdot \sin\left(\frac{2\pi}{1} n x\right) dx = 0$$

$$\text{so} \quad x^2 = \frac{1}{12} + \sum_{n=1}^{\infty} \frac{1}{(\pi n)^2} (-1)^n \cos(2\pi n x)$$

1. $x = 0$, $\quad \frac{\pi^2}{12} = \sum_{1}^{+\infty} \frac{1}{n^2} (-1)^{n+1}$

2. $x = \frac{1}{2}$, $\quad \frac{1}{4} - \frac{1}{12} = \sum_{1}^{+\infty} \frac{1}{n^2} \cdot \frac{1}{\pi^2} \cdot (-1)^{2n} = \frac{1}{\pi^2} \cdot \sum_{1}^{+\infty} \frac{1}{n^2}$

3. $\int_{-\frac{1}{2}}^{\frac{1}{2}} x^4 dx = \left(\frac{1}{12}\right)^2 \cdot 1 + \frac{1}{2} \cdot \sum_{n=1}^{\infty} \left[\frac{1}{(\pi n)^2} (-1)^n \right]^2$

$$\frac{1}{80} = \frac{1}{12^2} + \frac{1}{2} \cdot \frac{1}{\pi^4} \cdot \sum_{1}^{+\infty} \frac{1}{n^4} \qquad \therefore \frac{1}{90} \pi^4 = \sum_{1}^{\infty} \frac{1}{n^4}$$

2. $f(x) = \begin{cases} 0 & -\pi < x < 0 \\ \sin(x) & 0 \le x \le \pi \end{cases}$

$$a_0 = \frac{1}{2\pi}\int_0^{\pi} \sin(x)\,dx = \frac{1}{\pi},$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} \sin(x)\cdot\cos(nx)\,dx = \frac{1}{\pi}\int_0^{\pi} \frac{1}{2}\cdot\left(\sin((n+1)x) - \sin((n-1)x)\right)dx.$$

$$= \frac{1}{2\pi}\cdot\left\{\frac{1}{n+1}\left[1-(-1)^{n+1}\right] - \frac{1}{n-1}\left[1-(-1)^{n-1}\right]\right\}$$

$$= \begin{cases} 0 & n \text{ odd} \\ -\frac{2}{\pi}\cdot\frac{1}{n^2-1} & n \text{ even} \end{cases}$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} \sin(x)\cdot\sin(nx)\,dx = \frac{1}{\pi}\int_0^{\pi}\frac{1}{2}\left[\cos((n-1)x) - \cos((n+1)x)\right]dx$$

$$= \begin{cases} 0 & n > 1 \\ \frac{1}{2} & n = 1 \end{cases}$$

$$\therefore\ f(x) = \frac{1}{\pi} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cos(2nx) + \frac{1}{2}\sin(x)$$

$$x=0 \Rightarrow \frac{1}{2} = \sum_{n=1}^{\infty}\frac{1}{4n^2-1}$$

Parseval's Identity

$$\int_{-\pi}^{\pi} f^2(x)\,dx = \int_0^{\pi}\frac{1-\cos(2x)}{2}\,dx = \frac{\pi}{2} = 2\pi\cdot a_0^2 + \pi\cdot\sum_{1}^{\infty}(a_n^2+b_n^2)$$

$$= \frac{2}{\pi} + \frac{4}{\pi}\cdot\sum_{n=1}^{\infty}\frac{1}{(4n^2-1)^2} + \frac{\pi}{4},$$

$$\therefore\ \sum_{n=1}^{\infty}\frac{1}{(4n^2-1)^2} = \frac{\pi^2-8}{16}.$$

"Inhomogeneous ODE"

1. $x'' + 4x = \cos(t) + \sin(t)$. $x(0) = 1$, $x'(0) = 1$

A fundamental set of solution is $\{\cos(2t), \sin(2t)\}$.

A special solution is of the form.

$$x_{sp}(t) = a_0 + \sum a_n \cdot \cos(nt) + \sum b_n \cdot \sin(nt)$$

$$x_{sp}'' + 4x_{sp} = 4a_0 + \sum_1^\infty (4-n^2)\, a_n \cos(nt) + \sum_1^\infty (4-n^2)\, b_n \sin(nt)$$

$\therefore$ $a_1 = \frac{1}{3}$, $b_1 = \frac{1}{3}$, others are 0.

$$x_{sp} = \frac{1}{3} \cdot \cos(t) + \frac{1}{3} \sin(t)$$

$$x = A \cdot \cos(2t) + B \cdot \sin(2t) + \frac{1}{3} \cos(t) + \frac{1}{3} \sin(t).$$

with $x(0) = 1$, $x'(0) = 1$. we know. $A = \frac{2}{3}$. $B = \frac{1}{3}$.

2. $x'' + 2x' + x = \cos(t)$. $x(0) = 1$, $x(\pi) = 1$

Fundamental solution is $\{e^{-t}, t \cdot e^{-t}\}$.

A special solution by guess should be $x_{sp}(t) = a \cdot \cos(t) + b \cdot \sin(t)$

$$x_{sp}'' + 2x_{sp}' + x_{sp} = 2b\cos(t) - 2a\sin(t) = \cos(t)$$

So $x(t) = A \cdot e^{-t} + B \cdot t \cdot e^{-t} + \frac{1}{2} \sin(t)$

with $x(0) = A = 1$

$$x(\pi) = e^{-\pi} + B \cdot \pi \cdot e^{-\pi} = 1 \quad \Rightarrow B = \frac{e^{\pi} - 1}{\pi}$$

3. $x'' + x = t$.

fundamental solution is $\{\cos(t), \sin(t)\}$

A special solution by guess is $x_{sp} = t$.

General solution is $A\cos(t) + B\cdot\sin(t) + t$.

4. $x'' + 4x = t$

$A\cdot\cos(2t) + B\cdot\sin(2t) + \frac{t}{4}$

"Heat Equation"

2. $\begin{cases} u_t = 3u_{xx}, & t>0,\ 0\le x\le\pi \\ u(0,t) = u(\pi,t) = 0 \\ u(x,0) = 5\cdot\sin(x) + 2\cdot\sin(5x) & 0<x<\pi \end{cases}$

Ans: $\frac{T'(t)}{3T(t)} = -\lambda = \frac{X''(x)}{X(x)}$

by $u(0,t) = u(\pi,t) = 0$, we should choose $\lambda = n^2$.

and $X_n(x) = \sin(nx)$, $T_n(t) = e^{-3n^2t}$

$u(x,t) = c + \sum_{n=1}^{\infty} a_n\cdot e^{-3n^2t}\cdot\sin(nx)$

$u(x,0) = 5\cdot\sin(x) + 2\cdot\sin(5x) \Rightarrow u(x,t) = 5\cdot e^{-3t}\cdot\sin(x) + 2\cdot e^{-75t}\cdot\sin(5x)$

1. $\begin{cases} u_t = 2u_{xx} & t>0, \quad 0\le x\le 1 \\ u_x(0,t) = u_x(1,t) = 0 \\ u(x,0) = x(1-x) & 0<x<1 \end{cases}$

$$\frac{T'}{2T} = \frac{X''}{X} = -\lambda.$$

Solve $X'' + \lambda X = 0$ with $X'(0) = X'(1) = 0$

Suitable λ_n and corresponding X_n are

$$X_n(x) = \cos(n\pi \cdot x). \quad \lambda = (n\pi)^2.$$

So $u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n \cdot e^{-2(n\pi)^2 t} \cdot \cos(n\pi x)$

and $u(x,0) = x(1-x) = a_0 + \sum_{n=1}^{\infty} a_n \cdot e^{-2(n\pi)^2 t} \cdot \cos(n\pi x)$

How to compute the cosine series of $u(x,0)$?

Here's a difference between the example talked in the tutorial.

$\begin{cases} u_t = 2u_{xx} & t>0 \quad 0\le x<1 \\ u(0,t) = u(1,t) = 0 \\ u(x,0) = x(1-x) & 0<x<1 \end{cases}$

(Example in tutorial)

$\begin{cases} u_t = 2u_{xx} & t>0 \quad 0\le x<1 \\ u_x(0,t) = u_x(1,t) = 0 \\ u(x,0) = x(1-x) & 0<x<1 \end{cases}$

(Exercise).

In the note, it's written that $x(1-x) = \sum_{n \text{ odd}} \frac{8}{(n\pi)^3} \sin(n\pi x)$

So maybe some of you thoughe that has contradiction ?

$$x(1-x) = a_0 + \sum_{n=1}^{\infty} a_n \cdot \cos(n\pi x)$$

Well. actually the difference is due to the way we expand $x(1-x)$ $(0<x<1)$ to 2-periodic function $(x\in\mathbb{R})$.

To obtain $x(1-x) = \sum \frac{8}{(n\pi)^3} \sin(n\pi x)$, $x \in [0,1]$ a sine series.

we must notice several things.

① $\sum_{n=1}^{\infty} a_n \cdot \sin(n\pi x)$ is a 2-periodic function

(because $\sin(\pi x)$ is 2-periodic)

So $x(1-x)$ with $x \in [0,1]$ is not enough for computing Fourier series.

Thus. we need to extend it to an interval of length 2.

e.g. $f(x) = \begin{cases} x(1-x) & x \in [0,1] \\ \ldots\ldots & x \in [-1,0) \end{cases}$ or

$f(x) = \begin{cases} x(1-x) & x \in [0,1] \\ \ldots\ldots & x \in (1,2] \end{cases}$

② Since we hope $x(1-x) = \sum_{n=1}^{+\infty} a_n \cdot \sin(n\pi x)$,

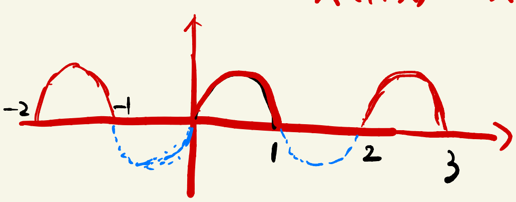
we should require the extension f to be odd. such that.

$\int f \cos(nx)\, dx = 0$.

Therefore. we extend f to be 2-periodic with its definition on.

$[-1, 1]$ being

$$f(x) = \begin{cases} x(1-x) & x \in [0,1] \\ x(1+x) & x \in [-1,0] \end{cases}$$



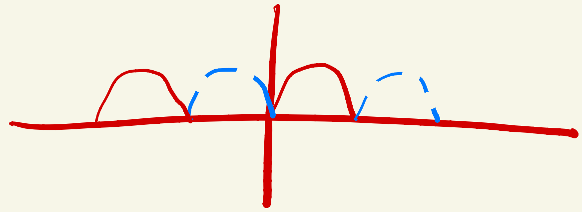
And $b_n = \frac{1}{1} \int_{-1}^{1} f(x) \cdot \sin(nx)\, dx = \frac{8}{(n\pi)^3}$

And for this exercise. we want to compute

$x(1-x) = a_0 + \sum_{n=1}^{\infty} a_n \cdot \cos(n\pi x)$ a cosine series

we need to extend $x(1-x)$ to be a 2-periodic even function

$$f(x) = \begin{cases} x(1-x) & x\in[0,1] \\ -x(1+x) & x\in[-1,0] \end{cases}$$



And $a_0 = \frac{1}{2}\int_{-1}^{1} f(x)\,dx = \int_0^1 x(1-x)\,dx = \frac{1}{6}$

$$a_n = \frac{1}{1}\cdot\int_{-1}^{1} f(x)\cdot\cos(n\pi x)\,dx = 2\cdot\int_0^1 x(1-x)\cos(n\pi x)\,dx$$

$$= 2\cdot\int_0^1 x\cdot\cos(n\pi x) - x^2\cos(n\pi x)\,dx$$

$$= 2\cdot\Big[\Big(\frac{1}{n\pi}\cdot x\cdot\sin(n\pi x)\Big|_0^1 - \frac{1}{n\pi}\int_0^1\sin(n\pi x)\,dx\Big) - \Big(\frac{1}{n\pi}x^2\cdot\sin(n\pi x)\Big|_0^1 - \frac{1}{n\pi}\int_0^1 2x\cdot\sin(n\pi x)\,dx\Big)\Big]$$

$$= 2\Big\{\frac{1}{(n\pi)^2}[1-(-1)^n] - \frac{2}{n\pi}\int_0^1 x\cdot\sin(n\pi x)\,dx\Big\}$$

$$= \frac{2\cdot[1-(-1)^n]}{(n\pi)^2} - \frac{4}{n\pi}\Big[\frac{-1}{n\pi}\cos(n\pi x)\cdot x\Big|_0^1 + \frac{1}{n\pi}\int_0^1\cos(n\pi x)\,dx\Big]$$

$$= \frac{2\cdot[1-(-1)^n] + 4(-1)^n}{(n\pi)^2} = \frac{2[1+(-1)^n]}{(n\pi)^2}$$

$$\therefore\ u(x,0) = x(1-x) = \frac{1}{6} + \sum_{n\text{ even}} \frac{4}{(n\pi)^2}\cdot\cos(n\pi x)$$

and $u(x,t) = \frac{1}{6} + \sum \frac{4}{\pi^2}\cdot\cos(2n\pi x)\cdot e^{-2(2n\pi)^2 t}$.

3. $\begin{cases} u_t = u_{xx} & t>0, \quad 0 \le x \le \pi \\ u(0,t) = u_x(\pi,t) = 0 \\ u(x,0) = 5 \cdot \sin(\frac{5}{2}x) & 0 < x < \pi . \end{cases}$

$$\frac{T'}{T} = -\lambda = \frac{X''}{X}$$

$u(0,t) = u_x(\pi,t) = 0 \Rightarrow$ solve eigenproblem. $X'' + \lambda X = 0$ with
$X(0) = 0,\ X'(\pi) = 0$

suitable eigenvalues and eigenfunctions are

$$\begin{cases} X_n(x) = \sin\left((\frac{1}{2}+n)x\right) \\ \lambda_n = (\frac{1}{2}+n)^2 \end{cases}$$

$$u(x,t) = a_0 + \Sigma a_n \cdot \sin\left((\tfrac{1}{2}+n)x\right) e^{-(\frac{1}{2}+n)^2 t}$$

$u(x,0) = 5 \cdot \sin(\frac{5}{2}x) \Rightarrow u(x,t) = 5 \cdot \sin(\frac{5}{2}x) \cdot e^{-(\frac{5}{2})^2 t}.$

4. $\begin{cases} u_t = a^2 u_{xx} & t>0, \quad 0 \le x \le \pi . \\ u(0,t) = T_1 \quad u(\pi,t) = T_2 \\ u(x,0) = f(x) & 0 < x < \pi . \end{cases}$

First. we define a trival solution $u_1(x,t) = ax+b$. then $(u_1)_t = 0 = (u_1)_{xx} \cdot a^2$
and $u_1(0,t) = b = T_1$
$u_1(\pi,t) = a \cdot \pi + T_1 = T_2$.

$u_1(x,t) = \frac{T_2 - T_1}{\pi} x + T_1$, we suppose solution $u(x,t)$ is of the form. $u(x,t) = u_1(x,t) + \underbrace{a_0 + \Sigma a_n \cdot X_n(x) \cdot T_n(t)}_{u_2(x,t)}$

and we just need to find $u_2(x,t)$ s.t.

$$\begin{cases} (u_2)_t = a^2 \cdot (u_2)_{xx} \\ u_2(0,t) = u_2(\pi,t) = 0 \\ u_2(x,0) = u(x,0) - u_1(x,0) = f(x) - \frac{T_2 - T_1}{\pi} x - T_1 \end{cases}$$

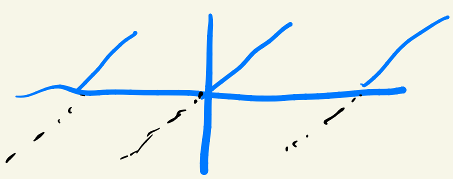
"Wave Equation"

1.
$$\begin{cases} u_{tt} = 2u_{xx} \\ u(0,t) = u(\pi,t) = 0 \\ u(x,0) = x \quad 0<x<\pi \\ u_t(x,0) = 0 \quad 0<x<\pi \end{cases}$$

$L=\pi$. $f(x) = x$. $g(x) = 0$. So we only need to compute

$$x \sim \sum_{n=1}^{\infty} C_n \cdot \sin(nx)$$

Since $\sum_1^{\infty} C_n \cdot \sin(nx)$ is 2π-periodic, we need to extend x $(x \in (0,\pi))$ to a 2π-periodic odd function f.



$f(x) = x$. for $x \in (-\pi, \pi)$

$$C_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \cdot \sin(nx)\,dx = \frac{2}{\pi} \cdot \int_0^{\pi} x \cdot \left(\frac{-1}{n}\cos(nx)\right)' dx$$

$$= \frac{2}{\pi}\left[x \cdot \left(\frac{-1}{n}\cos(nx)\right)\Big|_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos(nx)\,dx\right]$$

$$= \frac{2}{n}(-1)^{n+1}$$

$$\therefore u(x,t) = \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1} \cdot \cos(n \cdot \sqrt{2}\, t) \cdot \sin(nx)$$

2. $\begin{cases} u_{tt} = u_{xx} \\ u(0,t) = u(\pi,t) = 0 \\ u(x,0) = \sin(x) & x \in (0,\pi) \\ u_t(x,0) = \sin(x) & x \in (0,\pi) \end{cases}$

$u(x,t) = \sin(x) \cdot \cos(t) + \sin(x) \cdot \sin(t)$

3. $\begin{cases} u_{tt} = 0 & t > 0, \quad 0 \leq x \leq \pi \\ u(0,t) = u(\pi,t) = 0 \\ u(x,0) = \sin(2x) & 0 < x < \pi \\ u_t(x,0) = \sin(x) & 0 < x < \pi \end{cases}$

We still suppose $u(x,t) = \sum T_n(t) \cdot X_n(x) + \text{const.}$

with each $T_n(t) \cdot X_n(x)$ satisfying $\left(T_n(t) \cdot X_n(x)\right)_{tt} = 0.$

So $T_n(t) = a_n \cdot t + b_n$.

$u(x,t) = \sum (a_n t + b_n) X_n(x) + \text{const.}$

$u(0,t) = u(\pi,t) = 0. \Rightarrow X_n(0) = X_n(\pi) = 0, \quad \text{const} = 0.$

$u(x,0) = \sin(2x) \Rightarrow \sum b_n \cdot X_n(x) = \sin(2x)$

$u_t(x,0) = \sin(x) \Rightarrow \sum a_n \cdot X_n(x) = \sin(x)$

$\therefore X_n(x) = \sin(nx), \quad b_2 = 1. \quad a_1 = 1$

$u(x,t) = t \cdot \sin(x) + \sin(2x)$

4. $$\begin{cases} u_{tt} = a^2 u_{xx} - k \cdot u_t & t>0,\ 0 \le x \le 1 \\ u(0,t) = u(1,t) = 0 & \\ u(x,0) = f(x) & 0<x<1 \\ u_t(x,0) = 0, & 0<x<1 \end{cases}$$

$0<k<2\pi a$

Still. we suppose $u(x,t) = a_0 + \sum a_n \cdot X_n(x) \cdot T_n(t)$.

This time. $T_n''(t) \cdot X_n(x) = a^2 \cdot X_n''(x) \cdot T_n(t) - k \cdot X_n(x) \cdot T_n'(t)$

$$\frac{T_n''(t) + k T_n'(t)}{a^2 T_n(t)} = \frac{X_n''(x)}{X_n(x)} = -\lambda_n \quad (\lambda_n > 0)$$

$u(0,t) = u(1,t) = 0 \Rightarrow X_n(0) = X_n(1) = 0 \Rightarrow X_n(x) = \sin(n\pi x)$.

$\lambda_n = (n\pi)^2$

for T_n, we have $T_n'' + k \cdot T_n' + (n\pi a)^2 T_n = 0$.

According to The note 1's related conclusion to $ax'' + bx' + cx = 0$

since $k^2 - 4(n\pi a)^2 = k^2 - (2n\pi a)^2 < 0 \quad \forall n \ge 1$

by assumption $0<k<2\pi a$

Then suppose roots to $r^2 + k \cdot r + (n\pi a)^2 = 0$ are

$-k \pm i \cdot \beta_n$.

we have fundamental solutions of $T_n'' + k \cdot T_n' + (n\pi a)^2 T_n = 0$

are $e^{-kt} \cdot \cos(\beta_n t)$, $e^{-kt} \cdot \sin(\beta_n t)$

And $u(x,t) = \sum a_n \cdot e^{-kt} \cdot \cos(\beta_n t) \cdot \sin(n\pi x)$
$+ \sum b_n \cdot e^{-kt} \cdot \sin(\beta_n t) \sin(n\pi x)$

with $u(x,0) = \sum a_n \sin(n\pi x) = f(x)$ --- ①

$$u_t(x,0) = \sum a_n \cdot \left(-k \cdot e^{-kt} \cdot \cos(\beta_n t) - \beta_n \cdot e^{-kt} \cdot \sin(\beta_n t)\right)\Big|_{t=0} \sin(n\pi x)$$

$$+ \sum b_n \left(-k \cdot e^{-kt} \cdot \sin(\beta_n t) + \beta_n \cdot e^{-kt} \cdot \cos(\beta_n t)\right)\Big|_{t=0} \sin(n\pi x)$$

$$= \sum \left(b_n \beta_n - k \cdot a_n\right) \sin(n\pi x) = 0 \quad \text{---} \quad ②$$

By fourier expansion. we can firsely obtain a_n by ①

then b_n by ②